Production Economy

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Production: Firms

What is a firm?

- How is it managed/organized?
- What can it do?

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- How is it managed/organized?
- What can it do?

Neoclassical approach:

- A firm is a "black box" that transforms inputs into outputs.
- Firms are profit maximizing.

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- For example, firm can produce quantity q of commodity ℓ using $(z_1, \ldots, z_{\ell-1})$ commodities as inputs. That is,

$$y = (-z_1, \ldots, -z_{\ell-1}, q).$$

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Production set: $Y = \{(z_1, \ldots, z_\ell)\}$.
This set characterises the "input-output possibilities."

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- is closed;
- is strictly convex;
- satisfies free disposal: if $y \in Y$ and $y' \leq y$, then $y' \in Y$.

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$$\pi(p) := \max_{y \in Y} p \cdot y.$$

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- Optimal supply $y(p)$ is unique (due to strict convexity).
- The properties of the solution, and the corresponding profit, mirror those of utility maximization in consumer theory (see graphical illustration).

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Interpretation of goods and utility: some of the goods could be thought of as different types of labor.

- An agent can be endowed with his labor, which could be consumed by the agent as leisure (and it gives him utility), or it could be supplied as labor to firms, in order to earn money for other forms of consumption.

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Profit maximizing firm: At price $p = (p_1, ..., p_l) \gg 0$, firm f chooses a bundle $\hat{y}^f(p) \in Y^f$ to maximize its profit $p \cdot y^f(p)$. We denote the maximum profit at price p by $\pi^f(p)$.

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Utility maximizing agent: At price p, the agent a has an income of $p \cdot \omega^a + \sum_{f \in F} s^{fa} \pi^f(p)$. He maximizes $U^a(x)$ subject to

$$x \in B^a(p) = \{x \in \mathbb{R}^m_+ : p \cdot x \leq p \cdot \omega^a + \sum_{f \in F} s^{fa} \pi^f(p)\}.$$

We denote agent a's utility-maximizing choice by $\hat{x}^a(p)$.

Excess Demand

Excess demand: Given any market price p, the (aggregate) excess demand is

$$Z(p) = \sum_{a \in A} \hat{x}^a(p) - \left\{ \sum_{a \in A} \omega^a + \sum_{f \in F} \hat{y}^f(p) \right\}.$$

See graphical illustration.

Production Economy

Definition

The price vector $p^* \gg 0$ is a Walrasian equilibrium price if there is $\hat{y}^f(p^*)$ for each firm f and $\hat{x}^a(p^*)$ for each agent a such that $Z(p^*) = 0$.

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In the case where a good j is a pure factor, i.e., it does not give utility to anyone in the economy (e.g., raw materials like coal or timber), then $\sum_{a \in A} \hat{x}^a_j(p^*) = 0$ and so

$$\sum_{a \in A} \omega^a_j + \sum_{f \in F} \hat{y}^f_j(p^*) = 0.$$

So it can be part of the endowment of some agent, who will sell all of his endowment of j at price p^*_j, and it is then used by firms as an input (so $y^f_j < 0$ for some firm f).

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For an intermediate good k, which is a good created and traded as part of the production process but is not endowed to any agent and never consumed by consumers (e.g., steel, tactiles), we have

$$\sum_{f \in F} \hat{y}_k^f(p^*) = 0.$$

Production Economy: Example

n agents, 2 goods.

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where L is interpreted as the agent's consumption of his own labor, i.e., his leisure, and c is the level of consumption of the consumer good, of which there is just one.

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Firm shares: Agent a in the economy has a share s^a of this firm, so $\sum_{a \in A} s^a = 1$.

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$\Rightarrow$ optimal demand for labor at $L^* = p^2$;
$\Rightarrow$ optimal profit of the firm is $2p^2 - p^2 = p^2$.

Production Economy: Example

Optimal demand: The income of agent a's is $1 + s^a \cdot p^2$. Thus, his budget set is

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$$\sum_{a \in A} \frac{1 + s^a \cdot p^2}{2p} = \frac{n + p^2}{2p},$$

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Walrasian equilibrium price: Solving this equation we obtain $p = \sqrt{\frac{n}{3}}$.

Agent a's supply of labor is

$$1 - \frac{1}{2}\left(1 + \frac{s^a \cdot n}{3}\right).$$

Equilibrium Existence

Theorem

The excess demand function $Z : \mathbb{R}^{\ell}_{++} \to \mathbb{R}^{\ell}$ of the economy $\mathcal{E}$ (under assumption (P1), (P2), (P3)) has the following properties:

(1) *it is zero-homogenous,*

(2) *it obeys Walras' Law,*

(3) *it is continuous,*

(4) *it satisfies the boundary condition,*

(5) *it is bounded below.*

Remark: applies beyond exchange economy.

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Lemma

If the agents' utility functions are continuous, strictly monotone and strictly quasi-concave, and the firms' production sets are closed, strictly convex, bounded above, and a strictly positive aggregate consumption bundle is producible from the initial endowments, the excess demand function satisfies the above properties.

Productive Efficiency

Theorem

Suppose $p^* \gg 0$ is the Walrasian equilibrium price of a production economy and suppose that at this price firm f is producing $\hat{y}^f(p^*)$. Then there does not exist y^f (for each $f \in F$) such that

$$\sum_{f \in F} \hat{y}^f(p^*) < \sum_{f \in F} y^f. \tag{1}$$

In other words, there is productive efficiency at the Walrasian equilibrium (there is no other production that requires less inputs and produces more outputs).

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Proof: By way of contradiction, suppose (1) holds. Taking the inner product of both sides by $p^* \gg 0$, we obtain

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Suppose $p^* \gg 0$ is the Walrasian equilibrium price of a production economy and suppose that at this price firm f is producing $\hat{y}^f(p^*)$. Then there does not exist y^f *(for each $f \in F$) such that*

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Thus there is some firm $\tilde{f}$ such that

$$p^* \cdot \hat{y}^{\tilde{f}}(p^*) < p^* \cdot y^{\tilde{f}},$$

which means firm $\tilde{f}$ is not maximizing profit – a contradiction. **QED**

Pareto Efficiency with Production

An allocation $\{z^a\}_{a \in A}$ is said to be feasible if there is $y^f \in Y^f$ such that

$$\sum_{a \in A} \omega^a + \sum_{f \in F} y^f = \sum_{a \in A} z^a. \tag{2}$$

Definition

An allocation $\{x^a\}_{a \in A}$ is Pareto optimal if it is not Pareto dominated by another feasible allocation, i.e., there is no feasible allocation $\{z^a\}_{a \in A}$ such that $u^a(z^a) \geq u^a(x^a)$ for all agents a and at least one agent has *strictly* higher utility.

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Theorem (First Welfare Theorem)

Suppose all agents have monotone utility functions. Then every Walrasian allocation $\{x^a\}_{a\in A}$ *is Pareto optimal.*

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Proof: Suppose $\{z^a\}_{a\in A}$ is feasible (obeying (2)) and a Pareto improvement of $\{\hat{x}^a(p^*)\}_{a\in A}$.

Therefore, $U^a(z^a) \geq U^a(\hat{x}^a(p^*))$ for all $a \in A$ with a strict inequality for some agent $\tilde{a}$.

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Therefore, $U^a(z^a) \geq U^a(\hat{x}^a(p^*))$ for all $a \in A$ with a strict inequality for some agent $\tilde{a}$. By definition, $\hat{x}^{\tilde{a}}(p^*)$ maximizes agent $\tilde{a}$'s utility in a's budget set. So the bundle $z^{\tilde{a}}$ cannot be affordable to agent $\tilde{a}$, i.e.,

$$p^* \cdot z^{\tilde{a}} > p^* \cdot \omega^{\tilde{a}} + \sum_{f\in F} s^{f\tilde{a}} p^* \cdot \hat{y}^f(p^*).$$

Furthermore, for all agents in A,

$$p^* \cdot z^a \geq p^* \cdot \omega^a + \sum_{f\in F} s^{fa} p^* \cdot \hat{y}^f(p^*).$$

(This claim uses monotonicity; see proof of this theorem in exchange economies.)

Pareto efficiency with Production

Proof, continued: Summing across all agents and using the fact that $p^* \cdot \hat{y}^f(p^*) \geq p^* \cdot y^f$ for all f, we obtain

$$\begin{aligned} p^* \cdot \left[\sum_{a \in A} z^a\right] &> p^* \cdot \left[\sum_{a \in A} \omega^a\right] + p^* \cdot \left[\sum_{f \in F} \hat{y}^f(p^*)\right] \\ &\geq p^* \cdot \left[\sum_{a \in A} \omega^a\right] + p^* \cdot \left[\sum_{f \in F} y^f\right], \end{aligned}$$

which means that (2) is violated. **QED**

The Second Welfare Theorem

Production economy with transfers: Endowment of each agent $a \in A$:

- commodities ω^a;
- monetary transfer t^a.

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Theorem

Suppose that U^a is strongly monotone, strictly quasiconcave, and continuous for all a, and the production set Y^f is closed and strictly convex for all f. Then every Pareto optimal allocation is a Walrasian allocation with transfers.