General Equilibrium

Yingkai Li

EC5881 Semester 1, AY2024/25

Logistics

Instructor: Yingkai Li
Office: AS2 05-21
Office hour: Friday 3-4pm or by appointment.

Logistics

Instructor: Yingkai Li
Office: AS2 05-21
Office hour: Friday 3-4pm or by appointment.

Course Outline: General Equilibrium

1. Exchange Economy
2. Welfare Theorems
3. Production Economy

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Example: Alice and Bob, both of them have two apples and two bananas. Heterogeneity in preferences.

- Alice has utility 2 for the apple and utility 1 for the banana.
- Bob has utility 1 for the apple and utility 10 for the banana.
- Inefficient endowment.

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Example: Alice and Bob, both of them have two apples and two bananas. Heterogeneity in preferences.

- Alice has utility 2 for the apple and utility 1 for the banana.
- Bob has utility 1 for the apple and utility 10 for the banana.
- Inefficient endowment.

Two possible trades that could make both happier (Pareto improvement):

- Alice trades 2 bananas to Bob in exchange for 2 apples;

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Example: Alice and Bob, both of them have two apples and two bananas. Heterogeneity in preferences.

- Alice has utility 2 for the apple and utility 1 for the banana.
- Bob has utility 1 for the apple and utility 10 for the banana.
- Inefficient endowment.

Two possible trades that could make both happier (Pareto improvement):

- Alice trades 2 bananas to Bob in exchange for 2 apples;
- Alice trades 1 banana to Bob in exchange for 2 apples.

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Example: Alice and Bob, both of them have two apples and two bananas. Heterogeneity in preferences.

- Alice has utility 2 for the apple and utility 1 for the banana.
- Bob has utility 1 for the apple and utility 10 for the banana.
- Inefficient endowment.

Two possible trades that could make both happier (Pareto improvement):

- Alice trades 2 bananas to Bob in exchange for 2 apples;
- Alice trades 1 banana to Bob in exchange for 2 apples.

How people should trade for exchange items? What can be sustained in equilibrium in the exchange economy? What are the properties of the equilibria?

Exchange Economy

People exchange items in an economy for more efficient allocation of items.

Example: Alice and Bob, both of them have two apples and two bananas. Heterogeneity in preferences.

- Alice has utility 2 for the apple and utility 1 for the banana.
- Bob has utility 1 for the apple and utility 10 for the banana.
- Inefficient endowment.

Two possible trades that could make both happier (Pareto improvement):

- Alice trades 2 bananas to Bob in exchange for 2 apples;
- Alice trades 1 banana to Bob in exchange for 2 apples.

How people should trade for exchange items? What can be sustained in equilibrium in the exchange economy? What are the properties of the equilibria?

- use equilibrium price to exchange the items for efficient allocations.

Exchange Economy

Exchange economy: a finite set A of agents, ℓ commodities.

Exchange Economy

Exchange economy: a finite set A of agents, ℓ commodities.

For each agent $a \in A$:

- utility function $U^a : \mathbb{R}^\ell_+ \to \mathbb{R}$
- endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in $\mathbb{R}^\ell_+$.

Exchange Economy

Exchange economy: a finite set A of agents, ℓ commodities.

For each agent $a \in A$:

- utility function $U^a : \mathbb{R}^\ell_+ \to \mathbb{R}$
- endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in $\mathbb{R}^\ell_+$.

Assume U^a satisfies: (P1) continuous, (P2) strongly monotone, and (P3) strictly quasiconcave

Exchange Economy

Exchange economy: a finite set A of agents, ℓ commodities.

For each agent $a \in A$:

- utility function $U^a : \mathbb{R}^\ell_+ \to \mathbb{R}$
- endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in $\mathbb{R}^\ell_+$.

Assume U^a satisfies: (P1) continuous, (P2) strongly monotone, and (P3) strictly quasiconcave

Aggregate endowment:

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Exchange Economy

Exchange economy: a finite set A of agents, ℓ commodities.

For each agent $a \in A$:

- utility function $U^a : \mathbb{R}^\ell_+ \to \mathbb{R}$
- endowment $\omega^a = (\omega^a_1, \omega^a_2, ..., \omega^a_l)$ in $\mathbb{R}^\ell_+$.

Assume U^a satisfies: (P1) continuous, (P2) strongly monotone, and (P3) strictly quasiconcave

Aggregate endowment:

$$\bar{\omega} = \sum_{a \in A} \omega^a \gg 0.$$

Given market prices $p \in \mathbb{R}^\ell$, the income of agent a is $w^a = p \cdot \omega^a$.

- what are the demand of the agents given market prices and their income?

Recap on Demands

An economy with ℓ commodities

- consumption space is $\mathbb{R}^{\ell}_{+}$ (the positive orthant)
- utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$
- endowment/income/budget w
- price vector $p = (p_1, \ldots, p_\ell) \in \mathbb{R}^{\ell}_{++}$

Recap on Demands

An economy with ℓ commodities

- consumption space is $\mathbb{R}^\ell_+$ (the positive orthant)
- utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$
- endowment/income/budget w
- price vector $p = (p_1, \ldots, p_\ell) \in \mathbb{R}^\ell_{++}$

The budget set of the agent is

$$B(p, w) = \left\{ x \in \mathbb{R}^\ell_+ : p \cdot x \leq w \right\}.$$

Given price-budget pair (p, w), the demand is

$$x^* \in \operatorname*{argmax}_{x \in B(p,w)} U(x).$$

Recap on Demands

Suppose that the utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ is
(P1) continuous, (P2) strongly monotone, and (P3) strictly quasi-concave.

Then for any (p, w) in $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$, there exists a *unique* element x^* in $\arg\max_{x \in B(p,w)} U(x)$. Moreover, for any $(p, w) \gg 0$, the demand function $\bar{x}(p, w) = \text{argmax}_{x \in B(p,w)} U(x)$ has the following properties:

(a) it is continuous;

(b) it obeys the budget identity [i.e., $p \cdot \bar{x}(p, w) = w$];

(c) it is zero-homogeneous, [i.e. $\bar{x}(tp, tw) = \bar{x}(p, w)$ for any $t > 0$];

(d) it obeys the boundary condition: if $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and $I = \{i : \bar{p}_i = 0\}$ is nonempty, then

$$\sum_{i \in I} \bar{x}_i(p^n, w^n) \to \infty.$$

Excess Demand

For any agent a, given endowment ω and price p:

- demand of agent a at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Excess Demand

For any agent a, given endowment ω and price p:

- demand of agent a at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Define $\hat{x}^a : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$.
Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Excess Demand

For any agent a, given endowment ω and price p:

- demand of agent a at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Define $\hat{x}^a : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$.
Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Lemma

z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

Excess Demand

For any agent a, given endowment ω and price p:

- demand of agent a at price p is $\bar{x}^a(p, p \cdot \omega^a)$.

Define $\hat{x}^a : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ by $\hat{x}^a(p) = \bar{x}^a(p, p \cdot \omega^a)$.
Agent a's excess demand function is $z^a(p) = \hat{x}^a(p) - \omega^a$.

Lemma

z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any $\lambda > 0$, and $p \cdot z^a(p) = 0$ for all p.

The lemma holds since the demand function $\bar{x}^a(p, w)$ is zero-homogeneous and obeys the budget identity for any agent a.

Excess Demand

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

$$Z(p) = X(p) - \bar{\omega}.$$

Excess Demand

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously, since $Z(p) = \sum_{a \in A} z^a(p)$.

Excess Demand

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously, since $Z(p) = \sum_{a \in A} z^a(p)$.

Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

Excess Demand

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously, since $Z(p) = \sum_{a \in A} z^a(p)$.

Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

- existence of equilibrium price $p^* \gg 0$ such that market clears;

Excess Demand

Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p).$$

The aggregate excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ is given by

$$Z(p) = X(p) - \bar{\omega}.$$

Z is zero-homogeneous and obeys Walras' Law, $p \cdot Z(p) = 0$ for all p. Both inherited from z^a, obviously, since $Z(p) = \sum_{a \in A} z^a(p)$.

Fundamental Question: What conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?

- existence of equilibrium price $p^* \gg 0$ such that market clears;
- since Z is zero-homogeneous, if p^* is an equilibrium price so is λp^* for any $\lambda > 0$.

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
- Agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$, with endowment $\omega^B = (0, 1)$.

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
- Agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$, with endowment $\omega^B = (0, 1)$.

Suppose the price (p_1, p_2)

- agent A's income is $w^A = p_1$;

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
- Agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$, with endowment $\omega^B = (0, 1)$.

Suppose the price (p_1, p_2)

- agent A's income is $w^A = p_1$;
- agent A's demand function is $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$;

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
- Agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$, with endowment $\omega^B = (0, 1)$.

Suppose the price (p_1, p_2)

- agent A's income is $w^A = p_1$;
- agent A's demand function is $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$;
- agent A's demand in this economy given price p is

$$\hat{x}^A(p) = \left(\frac{1}{3}, \frac{2p_1}{3p_2}\right).$$

Exchange Economy: Example

Suppose economy has two agents, A and B, and two commodities:

- Agent A's utility function is $U^A(x_1, x_2) = \ln x_1 + 2\ln x_2$, with endowment $\omega^A = (1, 0)$;
- Agent B's utility function is $U^B(x_1, x_2) = 2\ln x_1 + \ln x_2$, with endowment $\omega^B = (0, 1)$.

Suppose the price (p_1, p_2)

- agent A's income is $w^A = p_1$;
- agent A's demand function is $\bar{x}^A(p, w) = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$;
- agent A's demand in this economy given price p is

$$\hat{x}^A(p) = \left(\frac{1}{3}, \frac{2p_1}{3p_2}\right).$$

Similarly, $\hat{x}^B(p) = \left(\frac{2p_2}{3p_1}, \frac{1}{3}\right)$. Therefore,

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

Exchange Economy: Example

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.

Exchange Economy: Example

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3} \right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.

Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Exchange Economy: Example

$$Z(p) = \left(-\frac{2}{3} + \frac{2p_2}{3p_1}, \frac{2p_1}{3p_2} - \frac{2}{3}\right).$$

Setting $Z_1(p) = 0$ we obtain $p_1 = p_2$.

Setting $Z_2(p) = 0$ we obtain $p_1 = p_2$.

Equilibrium price is (λ, λ) for any $\lambda > 0$.

Exchange Economy: Excess Demand

Theorem

The excess demand function $Z : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell$ of the economy $\mathcal{E}$ (under assumption (P1), (P2), (P3)) has the following properties:

(1) *it is zero-homogenous,*

(2) *it obeys Walras' Law,*

(3) *it is continuous,*

(4) *it satisfies the boundary condition,*

(5) *it is bounded below.*

Note: Clear that Z is bounded below since

$$Z(p) = X(p) - \bar{\omega} > -\bar{\omega}.$$

Illustration: Cobb-Douglas utilities.

- demand of agent a for commodity j is $\alpha_j \cdot \frac{w^a}{p_j}$ where $w^a = p \cdot \omega^a$.

Exchange Economy: Equilirbrium Existence

Theorem (Arrow and Debreu '54; McKenzie '59)

Suppose Z satisfies properties (1) to (5). Then there is $p^* \gg 0$ such that $Z(p^*) = 0$.

Properties such as (P1), (P2), (P3) are sufficient for equilibrium existence, but not necessary.

Exchange Economy: Equilirbrium Existence

Theorem (Arrow and Debreu '54; McKenzie '59)

Suppose Z satisfies properties (1) to (5). Then there is $p^* \gg 0$ such that $Z(p^*) = 0$.

Properties such as (P1), (P2), (P3) are sufficient for equilibrium existence, but not necessary.

Proof uses Kakutani's fixed point theorem, which generalizes Brouwer's fixed point theorem to correspondences.

Exchange Economy: Equilirbrium Existence

Theorem (Arrow and Debreu '54; McKenzie '59)

Suppose Z satisfies properties (1) to (5). Then there is $p^* \gg 0$ such that $Z(p^*) = 0$.

Properties such as (P1), (P2), (P3) are sufficient for equilibrium existence, but not necessary.

Proof uses Kakutani's fixed point theorem, which generalizes Brouwer's fixed point theorem to correspondences.

Brouwer's fixed point theorem is a (far-reaching) generalization of the intermediate value theorem.

Intermediate Value Theorem

Theorem (Intermediate value theorem)

Let f be a continuous function defined on some interval $[a, b]$. If $f(a)$ and $f(b)$ are of different signs, then there is $c \in [a, b]$ such that $f(c) = 0$.

Equilibrium Existence: Two Commodities

- Normalize $p_2 = 1$ and only consider p_1 (z is zero-homogeneous).

Equilibrium Existence: Two Commodities

- Normalize $p_2 = 1$ and only consider p_1 (z is zero-homogeneous).
- Equilibrium exists, i.e., $Z(p_1, 1) = 0$, if and only if there exists p_1 such that $z_1(p_1, 1) = 0$ (Exercise, by Walras' law).

Equilibrium Existence: Two Commodities

- Normalize $p_2 = 1$ and only consider p_1 (z is zero-homogeneous).
- Equilibrium exists, i.e., $Z(p_1, 1) = 0$, if and only if there exists p_1 such that $z_1(p_1, 1) = 0$ (Exercise, by Walras' law).
- $z_1(p_1', 1) < 0$ for sufficiently small p_1', and $z_1(p_1'', 1) > 0$ for sufficiently large p_1'' (by boundary condition & bounded below).

Equilibrium Existence: Two Commodities

- Normalize $p_2 = 1$ and only consider p_1 (z is zero-homogeneous).
- Equilibrium exists, i.e., $Z(p_1, 1) = 0$, if and only if there exists p_1 such that $z_1(p_1, 1) = 0$ (Exercise, by Walras' law).
- $z_1(p_1', 1) < 0$ for sufficiently small p_1', and $z_1(p_1'', 1) > 0$ for sufficiently large p_1'' (by boundary condition & bounded below).
- There exists p_1 such that $z_1(p_1, 1) = 0$ (by continuity and intermediate value theorem).

Additional Discussion on Price Formation

Question: how equilibrium prices are reached?

Additional Discussion on Price Formation

Question: how equilibrium prices are reached?

Utility functions are private information of agents.

- allocation according to market equilibrium prices in general are not incentive compatible.

Additional Discussion on Price Formation

Question: how equilibrium prices are reached?

Utility functions are private information of agents.

- allocation according to market equilibrium prices in general are not incentive compatible.

Potential justifications:

- large market assumption (a continuum of agents): each agent is so small such that changing their valuation cannot influence the market price.

Additional Discussion on Price Formation

Question: how equilibrium prices are reached?

Utility functions are private information of agents.

- allocation according to market equilibrium prices in general are not incentive compatible.

Potential justifications:

- large market assumption (a continuum of agents): each agent is so small such that changing their valuation cannot influence the market price.
- learning dynamics that converges to market equilibrium, e.g., tâtonnement [Arrow et al. '59; Gale '63; Cole and Fleischer '08]. This requires strong assumptions on the utility functions.

Additional Discussion on Price Formation

Question: how equilibrium prices are reached?

Utility functions are private information of agents.

- allocation according to market equilibrium prices in general are not incentive compatible.

Potential justifications:

- large market assumption (a continuum of agents): each agent is so small such that changing their valuation cannot influence the market price.
- learning dynamics that converges to market equilibrium, e.g., tâtonnement [Arrow et al. '59; Gale '63; Cole and Fleischer '08]. This requires strong assumptions on the utility functions.
- ...