Mechanism Design and Auctions

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EC5881 Semester 1, AY2024/25

Auctions

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

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- all-pay auction.

Design optimal mechanisms for maximizing the principal's payoff:

- welfare maximization;
- revenue maximization;
- consumer surplus maximization.

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- if $\max_{j \neq i} b_j \geq v_i$: bidder i does not gain by bidding higher to win;
- if $\max_{j \neq i} b_j < v_i$: bidder i does not gain by bidding lower since the payment won't decrease conditional on winning, and losing is worse.

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Remark: this is a dominant strategy equilibrium, where all agents maximize their utility (by reporting truthfully) regardless of the strategies of other agents.

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Question: what are the equilibrium bidding strategies.

- hard to guess directly in general.

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Verify: For each bidder i with value v_i, supposing that the other bidder j bids according to $b_j(v_j) = \frac{v_j}{2}$, the utility for bidding b_j is

$$\mathbf{E}_{v_j \sim U[0,1]}\left[(v_i - b_i) \cdot \mathbf{1}\left(b_i \geq \frac{v_j}{2}\right)\right] = \begin{cases} (v_i - b_i) \cdot 2b_i & b_i \leq \frac{1}{2}; \\ v_i - b_i & b_i > \frac{1}{2}. \end{cases}$$

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By FOC, utility $(v_i - b_i) \cdot 2b_i$ is maximized at $b_i = \frac{v_i}{2}$ for any $v_i \in [0,1]$.

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Guess: each bidder i bids $b_i(v_i) = \frac{2v_i}{3} - \frac{v_i}{6(v_i+1)}$.

Verify: exercise.

Revenue Comparison

Which auction has higher expected revenue? First-price auction or second-price auction?

Revenue Comparison

A sanity check: consider two agents with values drawn from $U[0,1]$.

- first-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}\left[\frac{1}{2}\cdot\max\{v_1,v_2\}\right]=\int_0^1\left(\int_{v_1}^1\frac{v_2}{2}\,\mathrm{d}v_2+\int_0^{v_1}\frac{v_1}{2}\,\mathrm{d}v_2\right)\mathrm{d}v_1=\frac{1}{3}.$$

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- second-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}[\min\{v_1,v_2\}]=\int_0^1\left(\int_{v_1}^1 v_1\,\mathrm{d}v_2+\int_0^{v_1} v_2\,\mathrm{d}v_2\right)\mathrm{d}v_1=\frac{1}{3}.$$

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Not a coincidence!

Mechanism Design

A single item, n agents (bidders).

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The principal designs a mechanism to maximize the objective function:

- social welfare: $\mathbf{E}[\sum_i v_i x_i]$
- revenue: $\mathbf{E}[\sum_i p_i]$
- consumer surplus: $\mathbf{E}[\sum_i v_i x_i - p_i]$

Welfare Maximization

Implementing the first best: allocating to the agent with highest value.

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Second-price auction is a special case of VCG auction.

VCG Mechanisms

Consider an allocation problem with n agents.

- general outcome space Ω;
- each agent i has private type θ_i;
- each agent i has utility $v_i(\omega, \theta_i) - p_i$.

Remark: it captures public projects, private allocations and externality in values.

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VCG mechanism:

- **allocation**: chooses outcome

$$\omega^* = \operatorname*{argmax}_{\omega \in \Omega} \sum_i v_i(\omega, \theta_i).$$

- **payment**: each agent i pays his externality on the welfare

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \geq 0.$$

VCG Mechanisms

Agent i's utility in VCG mechanism:

$$v_i(\omega^*, \theta_i) - \left(\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \right)$$
$$= \sum_j v_j(\omega^*, \theta_j) - \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) \geq 0.$$

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Agent i's utility is maximized by truthfully reporting his type to choose the allocation ω^* that maximizes the welfare.

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In the special case of single-item auction: item is allocated to the highest bidder

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VCG mechanism reduces to the second-price auction.

Revenue Maximization

It is impossible to implement the first revenue:

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Implementing the second best: design a mechanism that maximizes the expected revenue among all possible mechanisms.

Revelation Principle

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Definition (Revelation Mechanisms)

A revelation mechanism M is a static mechanism with allocation rule $x : V \to \{0,1\}^n$ and payment rule $p : V \to \mathbb{R}$ such that mechanism M is individually rational (IR) and incentive compatible (IC), i.e., $\forall i$, and $\forall v_i, v_i' \in V_i$,

$$\mathbf{E}_{v_{-i}\sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, \tag{IR}$$

$$\mathbf{E}_{v_{-i}\sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq \mathbf{E}_{v_{-i}\sim F_{-i}}\left[v_i \cdot x_i(v_i', v_{-i}) - p_i(v_i', v_{-i})\right]. \tag{IC}$$

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Revelation Principle [Myerson '81]: it is without loss to focus on revelation mechanisms.

Taxation Principle

Alternative ways for representing the mechanisms.

Definition (Menu Mechanisms)

For each agent i, the principal offers a menu $\{(x^{(k)}(v_{-i}), p^{(k)}(v_{-i})\}_{k\geq 0}$ to the agent. Each agent chooses the utility maximizing entry from the menu.

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incentive compatibility $\Leftrightarrow$ each agent chooses the utility maximizing entry

Interim Approach

Interim allocation: $x_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[x_i(v_i, v_{-i})]$.
Interim payment: $p_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[p_i(v_i, v_{-i})]$.

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Interim utility: $U_i(v_i) = v_i \cdot x_i(v_i) - p_i(v_i)$.

Incentive Compatibility

Lemma (Payoff Equivalence)

A revelation mechanism M is incentive compatible if and only if (1) the interim allocation $x_i(v_i)$ is weakly increasing in v_i for all i, and (2)

$$U_i(v_i) = U_i(0) + \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Formal argument: envelope theorem [Milgrom and Segal '02]

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Intuitive argument (see graphic illustration on board):

- incentive compatibility $\Leftrightarrow$ $U_i(v_i)$ is convex, with its derivative equal to $x_i(v_i)$.

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- incentive compatibility $\Leftrightarrow$ $U_i(v_i)$ is convex, with its derivative equal to $x_i(v_i)$.

The interim utility of the agents is uniquely determined by the interim allocation, up to an affine transformation of $U_i(0)$.

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Expected revenue:

$$\mathrm{Rev}(M) = \sum_i \mathbf{E}_{v_i \sim F_i}[p_i(v_i)] = \sum_i \mathbf{E}_{v_i \sim F_i}\left[v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0)\right].$$

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The expected revenue is uniquely determined by the interim allocations, up to an affine transformation of $\sum_i U_i(0)$.

- In symmetric environments, both first-price auction and second-price auction allocate to the highest value agent, and $U_i(0) = 0$ for all i.

Revenue Maximization

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Optimal revenue is maximized at $U_i(0) = 0$ for all i.

$$\begin{aligned}
\mathrm{Rev}(M) &= \sum_i \mathbf{E}_{v_i \sim F_i}\left[v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z\right] \\
&= \sum_i \mathbf{E}_{v_i \sim F_i}\left[\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right) \cdot x_i(v_i)\right] && \text{(Integration by parts)} \\
&= \mathbf{E}_{v \sim F}\left[\sum_i \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right) \cdot x_i(v_i, v_{-i})\right]. && \text{(Linearity of expectation)}
\end{aligned}$$

Virtual Value Maximization

Let $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ be the virtual value of agent i.

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Ideally, the optimal mechanism allocates the item to the agent with highest virtual value.

- is incentive compatibility satisfied? Not in general.

Virtual Value Maximization

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Theorem (Myerson '81)

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- recall that incentive compatibility requires monotonicity in allocations.

Question: what is the economic meaning of virtual value maximization?

Marginal Revenue Maximization

Let $q(v) = 1 - F(v)$

- $v(q)$ is defined as the value corresponds to q.
- $v(q)$ is also the market price such that the total demand is q.

Revenue curve $R(q)$: the revenue from serving the agents using a price with demand q.

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Virtual value maximization $\Leftrightarrow$ marginal revenue maximization [Bulow and Robert '89].

Regularity $\Leftrightarrow$ marginal revenue is higher for higher value agents [Bulow and Robert '89].

Revenue Optimal Auctions

Focus on symmetric environments with regular distributions.

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- item is not sold if all agents have values below the reserve price;
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Remark: the optimal reserve price v^* does not depend on the number of agents.

- it is also the optimal price in the single agent problem.

Additional Thinking

- Alternative approach for directly deriving marginal revenue maximization as the optimal mechanism. See [Bulow and Robert '89].
- Revenue optimal mechanism for irregular distributions: ironing [Myerson '81].
- Optimal mechanism for consumer surplus maximization. See [Hartline and Roughgarden '08].

First-price Auction

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

First-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays his bid.

Symmetric Environments

Consider symmetric environments, i.e., $F_i = F_j, \forall i, j$.

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By payoff equivalence, we have

$$b_i(v_i) = \frac{p_i(v_i)}{x_i(v_i)} = v_i - \frac{1}{x_i(v_i)} \cdot \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Uniform Distributions

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Uniqueness of Equilibria in First-price Auction

1. The constructed equilibrium is unique among the set of symmetric equilibria.
2. There does not exist any asymmetric equilibrium [Chawla and Hartline '13].

$\Rightarrow$ The constructed equilibrium is unique among all possible equilibria.

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Computing the equilibrium in asymmetric environments requires solving systems of differential equations in general [Plum '92; Kaplan and Zamir '12].

All-pay Auctions

A single item, n bidders.

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- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

Focus on symmetric environments.

All-pay Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- all agents pay their bids regardless winning or not.

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Equilibrium bids:

$$b_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z = v_i^2 - \int_0^{v_i} z\,\mathrm{d}z = \frac{v_i^2}{2}.$$