Games with Incomplete Information

Yingkai Li

EC5881 Semester 1, AY2024/25

Logistics

- Games with Incomplete Information
  - Bayesian Nash equilibrium (week 10)
  - Mechanism Design and Auctions (week 11)
- Comparative Statics (week 13)

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Office hour:

- 11am - 12pm Oct 30;
- 3pm - 4pm Nov 5;
- appointment by email if the above slots do not work for you.

Makeup class at 9am Oct 28!

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Bayesian Nash equilibrium / weak perfect Bayesian equilibrium

- Duopoly competition;
- Lemon market;
- Job market signaling;
- Evidence disclosure;
- First price auction.

Coordination with Incomplete Information

Two players coordinate on whether to watch a movie (M) or go to the park (P).

- The prior probability of "rain" is 0.1.
- Only the column player knows whether it will rain or not.

	M	P
M	$2, 1$	$0, 0$
P	$0, 0$	$1, 2$
Sunny		

	M	P
M	$2, 1$	$0, -5$
P	$-5, 0$	$-2, -2$
Rain		

Incomplete Information Games

A static game with incomplete information is denoted as $\Gamma_I = \left(N, (A_i)_{i\in N}, (u_i)_{i\in N}, (\Theta_i)_{i\in N}, \mu\right)$ where

- N is the set of players;
- A_i is the set of player i's actions;
- Θ_i is the set of player i's "types" where $\theta_i \in \Theta_i$ is private information of i;
- $u_i : A \times \Theta \to \mathbb{R}$ is player i's payoff function (where $A = \times_{i\in N} A_i$, and $\Theta = \times_{i\in N}\Theta_i$).
- $\mu(\theta)$ is the probability that a type profile $\theta \in \Theta$ occurs.

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Assume N, A, and Θ are all finite sets and $|N| = n$.

μ is called a common prior.

- Let μ_i denote the marginal distribution of μ on Θ_i, i.e., $\mu_i(\theta_i) \equiv \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_i, \theta_{-i})$.
- Let $\mu(\theta_{-i}|\theta_i)$ be the belief of agent i over θ_{-i} conditional on his type being θ_i.

Strategies and Bayesian Nash Equilibrium

A strategy of player i in Γ_I is a mapping $s_i : \Theta_i \to \Delta(A_i)$.

- s_i is a pure strategy if the mapping is deterministic, i.e., $s_i : \Theta_i \to A_i$. Let S_i be the set of pure strategies for i.

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Definition (BNE)

A strategy profile s is a Bayesian Nash Equilibrium if for any agent i and any type θ_i (such that $\mu_i(\theta_i) > 0$), for any action a_i^* in the support of $s_i(\theta_i)$, we have

$$a_i^* \in \operatorname*{argmax}_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \cdot \mathbf{E}_{a_{-i} \sim s_{-i}(\theta_{-i})}[u_i(a_i, a_{-i}, \theta)] .$$

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Intuition: treat each type as different agents and define Nash equilibrium similarly.

Strategic Games

Consider a complete information game $\Gamma_C = \left(N, \left(\widehat{A}_i\right)_{i\in N}, (\hat{u}_i)_{i\in N}\right)$ where for any $i \in N$,

- $\widehat{A}_i = S_i$;
- $\hat{u}_i(s) = \sum_{\theta\in\Theta} \mu(\theta) \cdot u_i\left(s_1(\theta_1), \ldots, s_n(\theta_n), \theta\right)$.

Note that $\widehat{A}_i$ is a finite set for all $i \in N$.

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Example for coordination game:

	MM	MP	PM	PP
M	$2, 1$	$1.8, 0.4$	0.2,0.1	0,-0.5
P	$-0.5, 0$	$-0.2, -0.2$	0.4,1.8	0.7, 1.6

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A strategy profile s is a Bayesian Nash equilibrium in Γ_I if and only if the induced action profile is a Nash equilibrium in Γ_C.

Ex ante vs Interim.

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Equivalent in finite games.

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For (2), not every mixed action in Γ_C is a valid mix strategy in Γ_I.

- Example: with probability $\frac{1}{2}$, choose action a given type θ and action a' given type θ', and with probability $\frac{1}{2}$, choose action a' given type θ and action a given type θ'.

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Indifferent for all players to consider strategies that induces the same marginal distribution over actions given any type.

- similar to the mix strategy vs behavioral strategy in repeated games.

Characterizing Bayesian Nash Equilibrium in Finite Games

1. Construct the corresponding strategic game Γ_C.
2. Characterize the set of Nash equilibrium in Γ_C.
3. Identify the corresponding Bayesian Nash equilibrium in Γ_I.

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Pure strategy equilibrium: $(M, MM), (P, PM)$
Mixed strategy equilibrium: exercise.

Equilibrium Computation

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Computing pure Bayesian Nash equilibria:

- in finite games: brute-force verification of all possible combinations;
- in infinite games: first-order methods.

Cournot Competition with Incomplete Information

Consider a Cournot duopoly model with incomplete information:

- 2 firms and 1 good.
- Each firm maximizes its own profits by simultaneously choosing a quantity to produce.
- Market price is $p = 1 - q_1 - q_2$.
- Firm 1's marginal cost is 0.
- Firm 2's marginal cost is 0 with probability θ and 0.5 with probability $1 - \theta$.
- Each firm knows only its own marginal cost and both are risk-neutral.

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Remark: this is a game with infinitely many actions.

- The definition of Bayesian Nash equilibrium extends easily but its existence is not always guaranteed.

Cournot Competition with Incomplete Information

Focus on pure Bayesian Nash equilibrium: given firm 1's quantity choice q_1,

- If firm 2's marginal cost is 0, then it solves

$$\max_{q_{2,L}} \left(1 - q_1 - q_{2,L}\right) q_{2,L}. \tag{1}$$

- If firm 2's marginal cost is 0.5, then it solves

$$\max_{q_{2,H}} \left(1 - q_1 - q_{2,H} - 0.5\right) q_{2,H}. \tag{2}$$

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Given firm 2's quantity choice $q_{2,L}, q_{2,H}$,

- Firm 1's problem should be

$$\max_{q_1} \theta \left(1 - q_1 - q_{2,L}\right) q_1 + (1 - \theta) \left(1 - q_1 - q_{2,H}\right) q_1 \tag{3}$$

Cournot Competition with Incomplete Information

Now derive FOCs from (1)-(3):

$$\begin{aligned} q_{2,L} &= \frac{1-q_1}{2}; \\ q_{2,H} &= \frac{1-q_1-0.5}{2}; \\ q_1 &= \frac{\theta\left(1-q_{2,L}\right)+(1-\theta)\left(1-q_{2,H}\right)}{2}. \end{aligned}$$

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\end{aligned}$$

The FOC method is valid since the maximization problems from (1)-(3) is concave.

Cournot Competition with Incomplete Information

Solutions:

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\begin{aligned}
q_1 &= \frac{1.5 - 0.5\theta}{3}; \\
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With higher probability of costly production for firm 2 (lower θ):

- higher quantity provided by firm 1 in equilibrium;
- lower quantity provided by firm 2 in equilibrium regardless of the cost type.

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Exercise: Does mixed Bayesian Nash equilibrium exist?

Extensive Form Games with Incomplete Information

Introduce nature as a non-strategic player.

- see illustration on board.

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Define Nash equilibrium / weak perfect Bayesian equilibrium (wPBE) in the usual sense.

Definition

(σ, μ) is a weak perfect Bayesian equilibrium if:

1. σ is sequentially rational given μ;
2. μ is derived from σ through Bayes' rule wherever possible.

Lemon Markets

Market for lemons [Akerlof '70]:

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Applications:

- used cars markets;
- insurance market;
- credit market.

Lemon Markets

Single seller, single buyer, single item with uncertain quality:

- quality $q \sim U[0, 1]$;
- seller value: $v(q) = q$;
- buyer value: $u(q) = \frac{3q}{2}$.
- utility functions given allocation x and transfer t:

$$V(x, t; q) = t - v(q) \cdot x, \quad U(x, t; q) = u(q) \cdot x - t.$$

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Remark: buyer always has a higher value than the seller given any quality q.

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- seller with quality $q \leq p$ is willing to sell the item;
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Only lemons ($q = 0$) can be traded in equilibrium.

Suppose there exists an equilibrium where the market price is $p > 0$:

- seller with quality $q \leq p$ is willing to sell the item;
- the expected quality conditional on seller willing to sell at price p is $\frac{p}{2}$;
- the expected value of the buyer conditional on seller willing to sell at price p is $\frac{3p}{4} < p$;
- no trade occurs (except lowest quality 0 that can be trade at price 0).

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Seller benefit from transparency in equilibrium:

- the equilibrium payoff of the seller can be improved by credibly disclosing her private information (e.g., by certification) in lemon's markets.

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- higher ability candidates have lower costs for acquiring higher education;
- intuition: higher education signals higher ability in equilibrium.

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A continuum of workers with two types $\theta_H > \theta_L$ [Spence '73].

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A worker has utility $w - c(e,\theta)$ for receiving wage w when providing education e.

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- Firms compete on wages to hire workers. In a competitive market, all firms offer a wage equal to the posterior expected value of the worker.

Equilibrium Analysis

Different types of equilibria:

- pooling equilibrium: both types are indistinguishable in equilibrium.
- separating equilibrium: types are separated in equilibrium.
- hybrid equilibrium: both pooling and separating exist.

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Incentive for θ_H: implied by incentive for θ_L.

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Exercise: other equilibria with different off path beliefs and education levels?

Job Market Signaling

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Separating equilibrium:

- higher type chooses higher education in equilibrium;
- strictly positive education occurs in equilibrium, even if it is not helpful for productivity, just to signal the ability.

Evidence Disclosure

In various markets, workers may have hard evidence that perfectly reveal their abilities.

- salary history;
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Other applications: In online platforms, consumers are given the rights to erase their data.

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Example:

- if no worker reveals the evidence, all workers receive a wage equals $\mathbf{E}[F]$;
- if only workers with ability above $\frac{1}{2}$ reveals the evidence, each worker with ability $\theta > \frac{1}{2}$ receives a wage equals θ, and each worker with ability $\theta \leq \frac{1}{2}$ receive a wage equals $\mathbf{E}\left[F \mid \theta \leq \frac{1}{2}\right]$.

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- the wage of the workers for no disclosure is $\mathbf{E}[\mu] < \bar{\theta}$;
- workers with ability $\theta \in (\mathbf{E}[\mu], \bar{\theta}]$ would deviate to disclosure, contradiction.

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- Rely on the assumption that disclosure is costless (relates to signaling if disclosure is costly).
- Regulation on voluntary disclosure, e.g., protecting the workers by preventing the share of salary history.