Mechanism Design and Auctions

Yingkai Li

EC5301

Auctions

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

Auctions

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

Understand the behavior of the agents in various auctions:

- first-price auction;
- second-price auction;
- all-pay auction.

Auctions

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

Understand the behavior of the agents in various auctions:

- first-price auction;
- second-price auction;
- all-pay auction.

Design optimal mechanisms for maximizing the principal's payoff:

- welfare maximization;
- revenue maximization;
- consumer surplus maximization.

Second-price Auction

Second-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays the second highest bid.

Second-price Auction

Second-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays the second highest bid.

Truthful bidding as equilibrium: all bidders maximize their utilities by bidding $b_i(v_i) = v_i$.

Second-price Auction

Second-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays the second highest bid.

Truthful bidding as equilibrium: all bidders maximize their utilities by bidding $b_i(v_i) = v_i$.

- if $\max_{j \neq i} b_j \geq v_i$: bidder i does not gain by bidding higher to win;

Second-price Auction

Second-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays the second highest bid.

Truthful bidding as equilibrium: all bidders maximize their utilities by bidding $b_i(v_i) = v_i$.

- if $\max_{j \neq i} b_j \geq v_i$: bidder i does not gain by bidding higher to win;
- if $\max_{j \neq i} b_j < v_i$: bidder i does not gain by bidding lower since the payment won't decrease conditional on winning, and losing is worse.

Second-price Auction

Second-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays the second highest bid.

Truthful bidding as equilibrium: all bidders maximize their utilities by bidding $b_i(v_i) = v_i$.

- if $\max_{j \neq i} b_j \geq v_i$: bidder i does not gain by bidding higher to win;
- if $\max_{j \neq i} b_j < v_i$: bidder i does not gain by bidding lower since the payment won't decrease conditional on winning, and losing is worse.

Remark: this is a dominant strategy equilibrium, where all agents maximize their utility (by reporting truthfully) regardless of the strategies of other agents.

First-price Auction

First-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays his bid.

First-price Auction

First-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays his bid.

Question: what are the equilibrium bidding strategies.

- hard to guess directly in general.

Example: Uniform Distribution

Two bidders. The value distribution $F_i = U[0,1]$ for all i.

Example: Uniform Distribution

Two bidders. The value distribution $F_i = U[0, 1]$ for all i.

Guess: each bidder i bids $b_i(v_i) = \frac{v_i}{2}$.

Example: Uniform Distribution

Two bidders. The value distribution $F_i = U[0,1]$ for all i.

Guess: each bidder i bids $b_i(v_i) = \frac{v_i}{2}$.

Verify: For each bidder i with value v_i, supposing that the other bidder j bids according to $b_j(v_j) = \frac{v_j}{2}$, the utility for bidding b_i is

$$\mathbf{E}_{v_j \sim U[0,1]}\left[(v_i - b_i) \cdot \mathbf{1}\left(b_i \geq \frac{v_j}{2}\right)\right] = \begin{cases} (v_i - b_i) \cdot 2b_i & b_i \leq \frac{1}{2}; \\ v_i - b_i & b_i > \frac{1}{2}. \end{cases}$$

Example: Uniform Distribution

Two bidders. The value distribution $F_i = U[0,1]$ for all i.

Guess: each bidder i bids $b_i(v_i) = \frac{v_i}{2}$.

Verify: For each bidder i with value v_i, supposing that the other bidder j bids according to $b_j(v_j) = \frac{v_j}{2}$, the utility for bidding b_i is

$$\mathbf{E}_{v_j \sim U[0,1]}\left[(v_i - b_i) \cdot \mathbf{1}\left(b_i \geq \frac{v_j}{2}\right)\right] = \begin{cases} (v_i - b_i) \cdot 2b_i & b_i \leq \frac{1}{2}; \\ v_i - b_i & b_i > \frac{1}{2}. \end{cases}$$

Obviously the utility of bidder i is maximized by bidding at most $\frac{1}{2}$.

Example: Uniform Distribution

Two bidders. The value distribution $F_i = U[0,1]$ for all i.

Guess: each bidder i bids $b_i(v_i) = \frac{v_i}{2}$.

Verify: For each bidder i with value v_i, supposing that the other bidder j bids according to $b_j(v_j) = \frac{v_j}{2}$, the utility for bidding b_i is

$$\mathbf{E}_{v_j \sim U[0,1]}\left[(v_i - b_i) \cdot \mathbf{1}\left(b_i \geq \frac{v_j}{2}\right)\right] = \begin{cases} (v_i - b_i) \cdot 2b_i & b_i \leq \frac{1}{2}; \\ v_i - b_i & b_i > \frac{1}{2}. \end{cases}$$

Obviously the utility of bidder i is maximized by bidding at most $\frac{1}{2}$.

By FOC, utility $(v_i - b_i) \cdot 2b_i$ is maximized at $b_i = \frac{v_i}{2}$ for any $v_i \in [0,1]$.

Example: Quadratic Distribution

Two bidders. The value distribution $F_i(v_i) = \frac{1}{2}\left(v_i^2 + v_i\right)$ for all i and $v_i \in [0,1]$.

Example: Quadratic Distribution

Two bidders. The value distribution $F_i(v_i) = \frac{1}{2}\left(v_i^2 + v_i\right)$ for all i and $v_i \in [0, 1]$.

Guess: each bidder i bids $b_i(v_i) = \frac{2v_i}{3} - \frac{v_i}{6(v_i+1)}$.

Verify: exercise.

Revenue Comparison

Which auction has higher expected revenue? First-price auction or second-price auction?

Revenue Comparison

A sanity check: consider two agents with values drawn from $U[0,1]$.

- first-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}\left[\frac{1}{2}\cdot\max\{\,v_1,v_2\,\}\right]=\int_0^1\left(\int_{v_1}^1\frac{v_2}{2}\,\mathrm{d}v_2+\int_0^{v_1}\frac{v_1}{2}\,\mathrm{d}v_2\right)\,\mathrm{d}v_1=\frac{1}{3}.$$

Revenue Comparison

A sanity check: consider two agents with values drawn from $U[0,1]$.

- first-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}\left[\frac{1}{2}\cdot\max\{v_1,v_2\}\right]=\int_0^1\left(\int_{v_1}^1\frac{v_2}{2}\,\mathrm{d}v_2+\int_0^{v_1}\frac{v_1}{2}\,\mathrm{d}v_2\right)\,\mathrm{d}v_1=\frac{1}{3}.$$

- second-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}[\min\{v_1,v_2\}]=\int_0^1\left(\int_{v_1}^1 v_1\,\mathrm{d}v_2+\int_0^{v_1} v_2\,\mathrm{d}v_2\right)\,\mathrm{d}v_1=\frac{1}{3}.$$

Revenue Comparison

A sanity check: consider two agents with values drawn from $U[0,1]$.

- first-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}\left[\frac{1}{2}\cdot\max\{v_1,v_2\}\right]=\int_0^1\left(\int_{v_1}^1\frac{v_2}{2}\,\mathrm{d}v_2+\int_0^{v_1}\frac{v_1}{2}\,\mathrm{d}v_2\right)\mathrm{d}v_1=\frac{1}{3}.$$

- second-price auction:

$$\mathbf{E}_{v_1,v_2\sim U[0,1]}[\min\{v_1,v_2\}]=\int_0^1\left(\int_{v_1}^1 v_1\,\mathrm{d}v_2+\int_0^{v_1}v_2\,\mathrm{d}v_2\right)\mathrm{d}v_1=\frac{1}{3}.$$

Not a coincidence!

Mechanism Design

A single item, n agents (bidders).

- each agent i has value $v_i \sim F_i$ with support $V_i \subseteq \mathbb{R}_+$;
- each agent i has utility $u_i = v_i x_i - p_i$.

Mechanism Design

A single item, n agents (bidders).

- each agent i has value $v_i \sim F_i$ with support $V_i \subseteq \mathbb{R}_+$;
- each agent i has utility $u_i = v_i x_i - p_i$.

The principal designs a mechanism to maximize the objective function:

- social welfare: $\mathbf{E}[\sum_i v_i x_i]$
- revenue: $\mathbf{E}[\sum_i p_i]$
- consumer surplus: $\mathbf{E}[\sum_i v_i x_i - p_i]$

Welfare Maximization

Implementing the first best: allocating to the agent with highest value.

Welfare Maximization

Implementing the first best: allocating to the agent with highest value.

Implementation by second-price auction:

- in equilibrium, all agents truthfully report their own values;
- highest value agent wins the item.

Welfare Maximization

Implementing the first best: allocating to the agent with highest value.

Implementation by second-price auction:

- in equilibrium, all agents truthfully report their own values;
- highest value agent wins the item.

Second-price auction is a special case of VCG auction.

VCG Mechanisms

Consider an allocation problem with n agents.

- general outcome space Ω;
- each agent i has private type θ_i;
- each agent i has utility $v_i(\omega, \theta_i) - p_i$.

Remark: it captures public projects, private allocations and externality in values.

VCG Mechanisms

Consider an allocation problem with n agents.

- general outcome space Ω;
- each agent i has private type θ_i;
- each agent i has utility $v_i(\omega, \theta_i) - p_i$.

Remark: it captures public projects, private allocations and externality in values.

VCG mechanism:

- **allocation**: chooses outcome

$$\omega^* = \operatorname*{argmax}_{\omega \in \Omega} \sum_i v_i(\omega, \theta_i).$$

VCG Mechanisms

Consider an allocation problem with n agents.

- general outcome space Ω;
- each agent i has private type θ_i;
- each agent i has utility $v_i(\omega, \theta_i) - p_i$.

Remark: it captures public projects, private allocations and externality in values.

VCG mechanism:

- **allocation**: chooses outcome

$$\omega^* = \operatorname*{argmax}_{\omega \in \Omega} \sum_i v_i(\omega, \theta_i).$$

- **payment**: each agent i pays his externality on the welfare

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \geq 0.$$

VCG Mechanisms

Agent i's utility in VCG mechanism:

$$v_i(\omega^*, \theta_i) - \left(\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \right)$$
$$= \sum_{j} v_j(\omega^*, \theta_j) - \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) \geq 0.$$

VCG Mechanisms

Agent i's utility in VCG mechanism:

$$\begin{aligned} &v_i(\omega^*, \theta_i) - \left(\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j) \right) \\ &= \sum_j v_j(\omega^*, \theta_j) - \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) \geq 0. \end{aligned}$$

Agent i's utility is maximized by truthfully reporting his type to choose the allocation ω^* that maximizes the welfare.

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

If i is the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the second highest bid;

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

If i is the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the second highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j) = 0$.

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

If i is the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the second highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j) = 0$.

If i is not the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the highest bid;

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

If i is the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the second highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j) = 0$.

If i is not the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j)$ is also the highest bid.

VCG Mechanisms

In the special case of single-item auction: item is allocated to the highest bidder

$$p_i(\theta) = \max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j) - \sum_{j \neq i} v_j(\omega^*, \theta_j).$$

If i is the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the second highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j) = 0$.

If i is not the highest bidder:

- $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega, \theta_j)$ is the highest bid;
- $\sum_{j \neq i} v_j(\omega^*, \theta_j)$ is also the highest bid.

VCG mechanism reduces to the second-price auction.

Revenue Maximization

It is impossible to implement the first revenue:

- if the principal allocates to the highest value agent and charges the payment equal to the value, agents has incentive to misreport a lower value.

Revenue Maximization

It is impossible to implement the first revenue:

- if the principal allocates to the highest value agent and charges the payment equal to the value, agents has incentive to misreport a lower value.

Implementing the second best: design a mechanism that maximizes the expected revenue among all possible mechanisms.

Revelation Principle

In general the mechanism designed by the principal can be complex.

- it may involve multiple rounds of communications among the agents.

Revelation Principle

In general the mechanism designed by the principal can be complex.

- it may involve multiple rounds of communications among the agents.

Definition (Revelation Mechanisms)

A revelation mechanism M is a static mechanism with allocation rule $x : V \to \{0, 1\}^n$ and payment rule $p : V \to \mathbb{R}$ such that mechanism M is individually rational (IR) and incentive compatible (IC), i.e., $\forall i$, and $\forall v_i, v_i' \in V_i$,

$$\mathbf{E}_{v_{-i} \sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, \tag{IR}$$

$$\mathbf{E}_{v_{-i} \sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq \mathbf{E}_{v_{-i} \sim F_{-i}}\left[v_i \cdot x_i(v_i', v_{-i}) - p_i(v_i', v_{-i})\right]. \tag{IC}$$

Revelation Principle

In general the mechanism designed by the principal can be complex.

- it may involve multiple rounds of communications among the agents.

Definition (Revelation Mechanisms)

A revelation mechanism M is a static mechanism with allocation rule $x : V \to \{0,1\}^n$ and payment rule $p : V \to \mathbb{R}$ such that mechanism M is individually rational (IR) and incentive compatible (IC), i.e., $\forall i$, and $\forall v_i, v_i' \in V_i$,

$$\mathbf{E}_{v_{-i} \sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq 0, \quad \text{(IR)}$$

$$\mathbf{E}_{v_{-i} \sim F_{-i}}[v_i \cdot x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] \geq \mathbf{E}_{v_{-i} \sim F_{-i}}\left[v_i \cdot x_i(v_i', v_{-i}) - p_i(v_i', v_{-i})\right]. \quad \text{(IC)}$$

Revelation Principle [Myerson '81]: it is without loss to focus on revelation mechanisms.

Taxation Principle

Alternative ways for representing the mechanisms.

Definition (Menu Mechanisms)

For each agent i, the principal offers a menu $\{(x^{(k)}(v_{-i}), p^{(k)}(v_{-i})\}_{k\geq 0}$ to the agent. Each agent chooses the utility maximizing entry from the menu.

Taxation Principle

Alternative ways for representing the mechanisms.

Definition (Menu Mechanisms)

For each agent i, the principal offers a menu $\{(x^{(k)}(v_{-i}), p^{(k)}(v_{-i})\}_{k\geq 0}$ to the agent. Each agent chooses the utility maximizing entry from the menu.

Taxation Principle [Mirrlees '73]: it is without loss to focus on menu mechanisms.

- given any revelation mechanism M, for each agent i, offer the menu that contains all contingent allocation-payment pairs for each $v_i \in V_i$.

Taxation Principle

Alternative ways for representing the mechanisms.

Definition (Menu Mechanisms)

For each agent i, the principal offers a menu $\{(x^{(k)}(v_{-i}), p^{(k)}(v_{-i})\}_{k\geq 0}$ to the agent. Each agent chooses the utility maximizing entry from the menu.

Taxation Principle [Mirrlees '73]: it is without loss to focus on menu mechanisms.

- given any revelation mechanism M, for each agent i, offer the menu that contains all contingent allocation-payment pairs for each $v_i \in V_i$.

incentive compatibility $\Leftrightarrow$ each agent chooses the utility maximizing entry

Interim Approach

Interim allocation: $x_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[x_i(v_i, v_{-i})]$.
Interim payment: $p_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[p_i(v_i, v_{-i})]$.

Interim Approach

Interim allocation: $x_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[x_i(v_i, v_{-i})]$.
Interim payment: $p_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[p_i(v_i, v_{-i})]$.

Menu mechanisms effectively offer menu of interim allocation-payment pairs to each agent.

Interim Approach

Interim allocation: $x_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[x_i(v_i, v_{-i})]$.
Interim payment: $p_i(v_i) = \mathbf{E}_{v_{-i} \sim F_{-i}}[p_i(v_i, v_{-i})]$.

Menu mechanisms effectively offer menu of interim allocation-payment pairs to each agent.

Interim utility: $U_i(v_i) = v_i \cdot x_i(v_i) - p_i(v_i)$.

Incentive Compatibility

Lemma (Payoff Equivalence)

A revelation mechanism M is incentive compatible if and only if (1) the interim allocation $x_i(v_i)$ is weakly increasing in v_i for all i, and (2)

$$U_i(v_i) = U_i(0) + \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Formal argument: envelope theorem [Milgrom and Segal '02]

Incentive Compatibility

Lemma (Payoff Equivalence)

A revelation mechanism M is incentive compatible if and only if (1) the interim allocation $x_i(v_i)$ is weakly increasing in v_i for all i, and (2)

$$U_i(v_i) = U_i(0) + \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Formal argument: envelope theorem [Milgrom and Segal '02]

Intuitive argument (see graphic illustration on board):

- incentive compatibility $\Leftrightarrow$ $U_i(v_i)$ is convex, with its derivative equal to $x_i(v_i)$.

Incentive Compatibility

Lemma (Payoff Equivalence)

A revelation mechanism M is incentive compatible if and only if (1) the interim allocation $x_i(v_i)$ is weakly increasing in v_i for all i, and (2)

$$U_i(v_i) = U_i(0) + \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Formal argument: envelope theorem [Milgrom and Segal '02]

Intuitive argument (see graphic illustration on board):

- incentive compatibility $\Leftrightarrow$ $U_i(v_i)$ is convex, with its derivative equal to $x_i(v_i)$.

The interim utility of the agents is uniquely determined by the interim allocation, up to an affine transformation of $U_i(0)$.

Revenue Equivalence

Interim payment:

$$p_i(v_i) = v_i \cdot x_i(v_i) - U_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0).$$

Revenue Equivalence

Interim payment:

$$p_i(v_i) = v_i \cdot x_i(v_i) - U_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0).$$

Expected revenue:

$$\mathrm{Rev}(M) = \sum_i \mathbf{E}_{v_i \sim F_i}[p_i(v_i)] = \sum_i \mathbf{E}_{v_i \sim F_i}\left[v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0)\right].$$

Revenue Equivalence

Interim payment:

$$p_i(v_i) = v_i \cdot x_i(v_i) - U_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0).$$

Expected revenue:

$$\mathrm{Rev}(M) = \sum_i \mathbf{E}_{v_i \sim F_i}[p_i(v_i)] = \sum_i \mathbf{E}_{v_i \sim F_i}\left[v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z - U_i(0)\right].$$

The expected revenue is uniquely determined by the interim allocations, up to an affine transformation of $\sum_i U_i(0)$.

- In symmetric environments, both first-price auction and second-price auction allocate to the highest value agent, and $U_i(0) = 0$ for all i.

Revenue Maximization

Individual rationality $\Rightarrow U_i(0) \geq 0$ for all i.

Revenue Maximization

Individual rationality $\Rightarrow U_i(0) \geq 0$ for all i.

Optimal revenue is maximized at $U_i(0) = 0$ for all i.

$$\begin{aligned}
\mathrm{Rev}(M) &= \sum_i \mathbf{E}_{v_i \sim F_i}\left[v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z\right] \\
&= \sum_i \mathbf{E}_{v_i \sim F_i}\left[\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right) \cdot x_i(v_i)\right] && \text{(Integration by parts)} \\
&= \mathbf{E}_{v \sim F}\left[\sum_i \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right) \cdot x_i(v_i, v_{-i})\right]. && \text{(Linearity of expectation)}
\end{aligned}$$

Virtual Value Maximization

Let $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ be the virtual value of agent i.

$$\mathrm{Rev}(M) = \mathbf{E}_{v\sim F}\left[\sum_i \phi_i(v_i) \cdot x_i(v_i, v_{-i})\right].$$

Virtual Value Maximization

Let $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ be the virtual value of agent i.

$$\text{Rev}(M) = \mathbf{E}_{v\sim F}\left[\sum_i \phi_i(v_i) \cdot x_i(v_i, v_{-i})\right].$$

Ideally, the optimal mechanism allocates the item to the agent with highest virtual value.

- is incentive compatibility satisfied? Not in general.

Virtual Value Maximization

Definition (Regularity [Myerson '81])

A distribution F is regular if the induced virtual value $\phi(v)$ is weakly increasing in v.

Virtual Value Maximization

Definition (Regularity [Myerson '81])

A distribution F is regular if the induced virtual value $\phi(v)$ is weakly increasing in v.

Theorem (Myerson '81)

If the valuation distribution is regular for all agents, the revenue optimal mechanism allocates the item to the agent with highest virtual value.

Virtual Value Maximization

Definition (Regularity [Myerson '81])

A distribution F is regular if the induced virtual value $\phi(v)$ is weakly increasing in v.

Theorem (Myerson '81)

If the valuation distribution is regular for all agents, the revenue optimal mechanism allocates the item to the agent with highest virtual value.

For regular distributions, by allocating the item to the agent with the highest virtual value, the resulting interim allocation is weakly increasing in values.

- recall that incentive compatibility requires monotonicity in allocations.

Virtual Value Maximization

Definition (Regularity [Myerson '81])

A distribution F is regular if the induced virtual value $\phi(v)$ is weakly increasing in v.

Theorem (Myerson '81)

If the valuation distribution is regular for all agents, the revenue optimal mechanism allocates the item to the agent with highest virtual value.

For regular distributions, by allocating the item to the agent with the highest virtual value, the resulting interim allocation is weakly increasing in values.

- recall that incentive compatibility requires monotonicity in allocations.

Question: what is the economic meaning of virtual value maximization?

Marginal Revenue Maximization

Let $q(v) = 1 - F(v)$

- $v(q)$ is defined as the value corresponds to q.
- $v(q)$ is also the market price such that the total demand is q.

Revenue curve $R(q)$: the revenue from serving the agents using a price with demand q.

- $R(q) \triangleq v(q) \cdot q$.

Marginal Revenue Maximization

Let $q(v) = 1 - F(v)$

- $v(q)$ is defined as the value corresponds to q.
- $v(q)$ is also the market price such that the total demand is q.

Revenue curve $R(q)$: the revenue from serving the agents using a price with demand q.

- $R(q) \triangleq v(q) \cdot q$.

Marginal revenue $R'(q)$: the marginal revenue by serving additional agents given current demand q.

- $R'(q) = v(q) - \frac{q}{f(v(q))} = \phi(v(q))$.

Marginal Revenue Maximization

Let $q(v) = 1 - F(v)$

- $v(q)$ is defined as the value corresponds to q.
- $v(q)$ is also the market price such that the total demand is q.

Revenue curve $R(q)$: the revenue from serving the agents using a price with demand q.

- $R(q) \triangleq v(q) \cdot q$.

Marginal revenue $R'(q)$: the marginal revenue by serving additional agents given current demand q.

- $R'(q) = v(q) - \frac{q}{f(v(q))} = \phi(v(q))$.

Virtual value maximization $\Leftrightarrow$ marginal revenue maximization [Bulow and Robert '89].

Marginal Revenue Maximization

Let $q(v) = 1 - F(v)$

- $v(q)$ is defined as the value corresponds to q.
- $v(q)$ is also the market price such that the total demand is q.

Revenue curve $R(q)$: the revenue from serving the agents using a price with demand q.

- $R(q) \triangleq v(q) \cdot q$.

Marginal revenue $R'(q)$: the marginal revenue by serving additional agents given current demand q.

- $R'(q) = v(q) - \frac{q}{f(v(q))} = \phi(v(q))$.

Virtual value maximization $\Leftrightarrow$ marginal revenue maximization [Bulow and Robert '89].

Regularity $\Leftrightarrow$ marginal revenue is higher for higher value agents [Bulow and Robert '89].

Revenue Optimal Auctions

Focus on symmetric environments with regular distributions.

Revenue Optimal Auctions

Focus on symmetric environments with regular distributions.

Optimal mechanism allocates the item to the agent with the highest value/virtual value/marginal revenue if the highest virtual value is non-negative.

Revenue Optimal Auctions

Focus on symmetric environments with regular distributions.

Optimal mechanism allocates the item to the agent with the highest value/virtual value/marginal revenue if the highest virtual value is non-negative.

Optimal mechanism: second-price auction with anonymous reserve v^*

- item is not sold if all agents have values below the reserve price;
- v^* is the cutoff value with zero virtual value.

Revenue Optimal Auctions

Focus on symmetric environments with regular distributions.

Optimal mechanism allocates the item to the agent with the highest value/virtual value/marginal revenue if the highest virtual value is non-negative.

Optimal mechanism: second-price auction with anonymous reserve v^*

- item is not sold if all agents have values below the reserve price;
- v^* is the cutoff value with zero virtual value.

Remark: the optimal reserve price v^* does not depend on the number of agents.

- it is also the optimal price in the single agent problem.

Additional Thinking

- Alternative approach for directly deriving marginal revenue maximization as the optimal mechanism. See [Bulow and Robert '89].
- Revenue optimal mechanism for irregular distributions: ironing [Myerson '81].
- Optimal mechanism for consumer surplus maximization. See [Hartline and Roughgarden '08].

First-price Auction

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

First-price Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- winner pays his bid.

Symmetric Environments

Consider symmetric environments, i.e., $F_i = F_j, \forall i, j$.

Symmetric Environments

Consider symmetric environments, i.e., $F_i = F_j, \forall i, j$.

In symmetric environments, there always exists a symmetric equilibrium.

- $x_i(v_i) = \prod_{j \neq i} F_j(v_i), \forall i$.

Symmetric Environments

Consider symmetric environments, i.e., $F_i = F_j, \forall i, j$.

In symmetric environments, there always exists a symmetric equilibrium.

- $x_i(v_i) = \prod_{j \neq i} F_j(v_i), \forall i$.

By payoff equivalence, we have

$$b_i(v_i) = \frac{p_i(v_i)}{x_i(v_i)} = v_i - \frac{1}{x_i(v_i)} \cdot \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Interim allocation: $x_i(v_i) = F(v_i) = v_i$.

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Interim allocation: $x_i(v_i) = F(v_i) = v_i$.

Equilibrium bids:

$$b_i(v_i) = v_i - \frac{1}{x_i(v_i)} \cdot \int_0^{v_i} x_i(z)\,\mathrm{d}z = v_i - \frac{1}{v_i} \cdot \int_0^{v_i} z\,\mathrm{d}z = \frac{v_i}{2}.$$

Uniqueness of Equilibria in First-price Auction

1. The constructed equilibrium is unique among the set of symmetric equilibria.
2. There does not exist any asymmetric equilibrium [Chawla and Hartline '13].

$\Rightarrow$ The constructed equilibrium is unique among all possible equilibria.

Asymmetric Environments

Consider asymmetric environments, i.e., $\exists i \neq j$ such that $F_i \neq F_j$.

Asymmetric Environments

Consider asymmetric environments, i.e., $\exists i \neq j$ such that $F_i \neq F_j$.

Even guess the interim allocation in equilibrium can be challenging.

Asymmetric Environments

Consider asymmetric environments, i.e., $\exists i \neq j$ such that $F_i \neq F_j$.

Even guess the interim allocation in equilibrium can be challenging.

Computing the equilibrium in asymmetric environments requires solving systems of differential equations in general [Plum '92; Kaplan and Zamir '12].

All-pay Auctions

A single item, n bidders.

- each bidder i has value $v_i \sim F_i$;
- each bidder i has utility $u_i = v_i x_i - p_i$.

Assume distributions F_i are continuous for simplicity.

Focus on symmetric environments.

All-pay Auction: Each bidder i place a bid $b_i \geq 0$ in the auction.

- highest bidder wins where ties are broken uniform randomly;
- all agents pay their bids regardless winning or not.

All-pay Auctions

In symmetric environments, there always exists a symmetric equilibrium.

- $x_i(v_i) = \prod_{j \neq i} F_j(v_i), \forall i.$

All-pay Auctions

In symmetric environments, there always exists a symmetric equilibrium.

- $x_i(v_i) = \prod_{j \neq i} F_j(v_i), \forall i.$

By payoff equivalence, we have

$$b_i(v_i) = p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z.$$

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Interim allocation: $x_i(v_i) = F(v_i) = v_i$.

Uniform Distributions

Consider the simple case with two agents where F is uniform in $[0, 1]$.

Interim allocation: $x_i(v_i) = F(v_i) = v_i$.

Equilibrium bids:

$$b_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z = v_i^2 - \int_0^{v_i} z\,\mathrm{d}z = \frac{v_i^2}{2}.$$